CRITICAL SETS OF EIGENFUNCTIONS AND YAU CONJECTURE

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Abstract. S.T.Yau posed a conjecture that the number of critical points of the k-th eigenfunction on a compact Riemannian manifold (strictly) increases with k. As a counterexample, Jakobson and Nadirashvili constructed a metric on 2-torus such that the eigenvalues tend to infinity whereas the number of critical points remains a constant. The present paper finds several interesting eigenfunctions on the minimal isoparametric hypersurface M^n of FKM-type in $S^{n+1}(1)$, giving a series of counterexamples to Yau conjecture. More precisely, the three eigenfunctions on M^n correspond to eigenvalues n, $2n$ and $3n$, while their critical sets consist of 8 points, a submanifold and 8 points, respectively. On one of its focal submanifolds, a similar phenomenon occurs. However, it is possible that Yau conjecture holds true for a generic metric.

1. Introduction

Eigenvalues of Laplacian are very important intrinsic invariants, which reflect the geometry of manifolds very precisely. Unfortunately, there are few manifolds whose eigenvalues are clearly known, not to mention the eigenfunctions. The numbers of critical points of eigenfunctions are even more difficult to determine. However, as S.T.Yau pointed out, this number is closely related to many important questions, which makes it worthy of being studied extensively. In this regard, S.T.Yau [Yau] posed a conjecture that the number of critical points of the k-th eigenfunction on a compact Riemannian manifold (strictly) increases with k.

As a counterexample, Jakobson and Nadirashvili [JN] constructed a metric on a 2-dimensional torus and a sequence of eigenfunctions such that the corresponding eigenvalues go to infinity while the number of critical points remains bounded, a constant in fact. But in some senses, their example is not a virtual denial to Yau conjecture, since one might expect that Yau conjecture still hold true in the sense of "non-decreasing".

In the present paper, by taking advantage of a natural concept—isoparametric hypersurface, we find an isoparametric function, which is an eigenfunction on the minimal isoparametric hypersurface M^n of FKM-type in $S^{n+1}(1)$. Combining with the other

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two well-known eigenfunctions, it constitutes a series of counterexamples of Yau conjecture in a strict sense. Similarly, another isoparametric function (an eigenfunction in fact) expressed in the same form arises in one of the focal submanifolds of M^n mentioned before. This gives rise to another series of counterexamples of Yau conjecture. However, our metric is quite special, it is possible that Yau conjecture holds true for a generic metric. As is well known, K. Uhlenbeck has shown in 1976 that on a compact Riemannian manifold, when the metric is generic, the eigenvalues of the Laplacian are simple and the associated eigenfunctions are Morse functions.

One of the main results of the present paper is the following:

Theorem 1.1. *Let M^n be the minimal isoparametric hypersurface of FKM-type in the unit sphere $S^{n+1}(1)$. Then there exist three eigenfunctions φ_1, φ_2 and φ_3 defined on M^n, corresponding to eigenvalues n, $2n$ and $3n$, whose critical sets consist of 8 points, a submanifold and 8 points, respectively. For specific, φ_1 and φ_3 are both Morse functions; φ_2 is an isoparametric function on M^n, whose critical set $C(\varphi_2)$ is:*

$$C(\varphi_2) = N_+ \cup N_-, \quad \dim N_+ = \dim N_- = n - m \ (1 \leq m < n), \tag{1}$$

where the number m will be introduced in the definition of FKM-type.

Remark 1.1. The Morse number (the minimal number of critical points of all Morse functions) of a compact isoparametric hypersurface in the unit sphere is equal to $2g$ (*cf.* [CR]).

Firstly, to clarify notations, we denote the Laplacian on an n-dimensional compact manifold M^n by $\Delta f = \mathrm{div}\nabla f$, and say λ_k its k-th eigenvalue with multiplicity ($\lambda_0 = 0 < \lambda_1 < \lambda_2 < ...$) if $\Delta f_k + \lambda_k f_k = 0$ for some $f_k : M^n \to \mathbb{R}$. Correspondingly, f_k is called the k-th eigenfunction. The present paper is mainly concerned with the number of critical points of the eigenfunction f_k.

Recall that a hypersurface M^n in a Riemannian manifold $\widetilde{M}^{n+1}$ is *isoparametric* if it is a level hypersurface of an isoparametric function f on $\widetilde{M}^{n+1}$, that is, a non-constant smooth function $f : \widetilde{M}^{n+1} \to \mathbb{R}$ satisfying:

$$\begin{cases} |\widetilde{\nabla} f|^2 = b(f) \\ \quad \widetilde{\triangle} f \ = a(f) \end{cases} \tag{2}$$

where b and a are smooth and continuous functions on $\mathbb{R}$, respectively.

In this meaning, the *focal varieties* are the preimages of the global maximum and minimum values of f, which we denote by M_+ and M_-, respectively. They are in fact both minimal submanifolds of $\widetilde{M}^{n+1}$ with codimensions $m_+ + 1$ and $m_- + 1$ in $\widetilde{M}^{n+1}$, respectively(*cf.* [Wan], [GT1]).

As asserted by Élie Cartan, an isoparametric hypersurface in the unit sphere is indeed a hypersurface with constant principal curvatures. An elegant result of Münzner

states that the number g of distinct principal curvatures must be $1, 2, 3, 4$ or 6. Up to now, the isoparametric hypersurfaces with $g = 1, 2, 3, 6$ are completely classified (*cf.* [DN] and [Miy]). For isoparametric hypersurfaces with $g = 4$, Cecil-Chi-Jensen ([CCJ]), Immervoll ([Imm]) and Chi ([Chi]) proved a far reaching result that they are just of FKM-type except the cases $(m_+, m_-) = (2, 2),\ (4, 5)$ and $(7, 8)$.

From now on, we are specifically concerned with the isoparametric hypersurfaces of FKM-type $S^{n+1}(1)$ with four distinct principal curvatures. For a symmetric Clifford system $\{P_0, ..., P_m\}$ on $\mathbb{R}^{2l}$, *i.e.*, P_i's are symmetric matrices satisfying $P_iP_j + P_jP_i = 2\delta_{ij}I_{2l}$, the *FKM-type isoparametric hypersurfaces* are level hypersurfaces of $f := F|_{S^{2l-1}}$ with F defined by Ferus, Karcher and Münzner (cf. [FKM]):

$$\begin{aligned} &F : \mathbb{R}^{2l} \to \mathbb{R} \\ &F(x) = |x|^4 - 2\sum_{\alpha=0}^{m} \langle P_\alpha x, x\rangle^2 \end{aligned} \tag{3}$$

The pairs (m_+, m_-) of the FKM-type are $(m, l-m-1)$, provided $m > 0$ and $l-m-1 > 0$, where $l = k\delta(m)$ $(k = 1, 2, 3, ...)$, $\delta(m)$ is the dimension of an irreducible module of the Clifford algebra C_{m-1}, which we list below:

m	1	2	3	4	5	6	7	8	$\cdots$ m+8
$\delta(m)$	1	2	4	4	8	8	8	8	$16\delta(m)$

We now fix M^n to be the minimal isoparametric hypersurface of FKM-type in $S^{n+1}(1)$, and f to be $f := F|_{S^{2l-1}}$ with F defined in (3). Choosing a point $q_1 \in S^{n+1}(1)\backslash\{M_+, M_-, M^n\}$, we define three eigenfunctions φ_1, φ_2 (following [Sol]) and φ_3 as follows:

$$\begin{aligned} &\varphi_1 : M^n \to \mathbb{R}, \qquad &&\varphi_2 : M^n \to \mathbb{R} \qquad &&\varphi_3 : M^n \to \mathbb{R} \\ &x \mapsto \langle x, q_1\rangle, &&x \mapsto \langle Px, x\rangle &&x \mapsto \langle \xi(x), q_1\rangle \end{aligned} \tag{4}$$

where ξ is a unit normal vector field on M^n; $P \in \Sigma := \Sigma(P_0, ..., P_m)$, the unit sphere in $Span\{P_0, ..., P_m\}$, which is called *the Clifford sphere* (*cf.* [FKM]).

Remark 1.2. The authors proved recently that the first eigenvalue of the closed minimal isoparametric hypersurface M^n in $S^{n+1}(1)$ is just n (*cf.* [TY]). As a corollary, the coordinate function restricted on M^n is the first eigenfunction. The function φ_1 above is a special case.

With all the preconditions, a direct verification reveals that the eigenvalues corresponding to φ_1, φ_2 and φ_3 are n, $2n$ and $3n$, respectively. Moreover, with our choice of $q_1 \in S^{n+1}(1)\backslash\{M_+, M_-, M^n\}$, a simple application of isoparametric geometry shows that φ_1 and φ_3 are both Morse functions with $2g = 8$ critical points. The more fascinating result is that φ_2 is indeed an isoparametric function on M^n, thus by virtue of

[Wan], the critical set of φ_2 are just the union of its focal submanifolds N_+ and N_-. For the proof of Theorem 1.1, we need the following lemma, in which we also show an interesting phenomenon occurring in the improper case (*cf.* [GT2]):

Lemma 1.1. *For focal submanifolds N_+ and N_- of φ_2 on M^n, we have diffeomorphisms:*

$$N_+ \underset{diff.}{\cong} N_- \underset{diff.}{\cong} M_+ = \{x \in S^{n+1}(1) \mid \langle P_0x, x\rangle = \langle P_1x, x\rangle = \cdots = \langle P_mx, x\rangle\}.$$

Particularly, in the improper case, i.e. $m = 1$, each level (isoparametric) hypersurface of φ_2 is minimal.

As we stated before, another series of counterexamples of Yau conjecture appear on the focal submanifold $M_- := f^{-1}(-1)$ with dimension $l + m - 1$. In a similar way, we define two eigenfunctions ω_1, ω_2 on M_-:

$$\begin{aligned} \omega_1 : M_- &\to \mathbb{R} & \qquad \omega_2 : M_- &\to \mathbb{R} \\ x &\mapsto \langle Px, x\rangle & x &\mapsto \langle x, q_2\rangle, \end{aligned} \tag{5}$$

where $P \in \Sigma = \Sigma(P_0, P_1, ..., P_m)$, $q_2 \in S^{n+1}(1)\backslash\{M_+, M_-\}$. Correspondingly, we have the following theorem:

Theorem 1.2. *Let $M_- := f^{-1}(-1)$ be the focal submanifold of FKM-type in the unit sphere $S^{n+1}(1)$. Then there exist two eigenfunctions ω_1 and ω_2 defined on M_-, corresponding to eigenvalues $4m$ and $l + m - 1$, whose critical sets consist of a submanifold and 4 points, respectively. For specific, ω_2 is a Morse function; ω_1 is an isoparametric function on M_-, whose critical set $C(\omega_1)$ is:*

$$C(\omega_1) = V_+ \cup V_-, \quad \dim V_+ = \dim V_- = l - 1. \tag{6}$$

Remark 1.3. The Morse number of each focal submanifold of a compact isoparametric hypersurface in the unit sphere is equal to g (*cf.* [CR]).

For the proof of Theorem 1.2, we need the following:

Lemma 1.2. *For focal submanifolds V_+ and V_- of ω_1 on M_-, we have isometries:*

$$V_+ \underset{isom.}{\cong} V_- \underset{isom.}{\cong} S^{l-1}(1).$$

Particularly, in the improper case, i.e. $m = 1$, each level (isoparametric) hypersurface of ω_1 is minimal.

Comparing with the values of $\delta(m)$ in the previous table, we observe that $4m < l + m - 1$ at most cases. More precisely, $4m < l + m - 1$ as long as $k \geq 5$ and $m \leq 9$; $4m < l + m - 1$ holds true for any k when $m \geq 10$. Therefore, with an appropriate

choice of k, we can always make eigenfunctions ω_1 and ω_2 another counterexample of Yau conjecture.

Bearing these examples in mind, we would like to pose the following:

Conjecture: For a generic metric on a compact manifold M, the number of critical points of the first eigenfunction (must be a Morse function, according to Uhlenbeck) is equal to the Morse number of M!

2. COUNTEREXAMPLES ON M^n

This section will be committed to proving Theorem 1.1 on the minimal isoparametric hypersurface M^n of FKM-type in $S^{n+1}(1)$. At first, we denote the connections and Laplacians on M^n, $S^{n+1}(1)$ and $\mathbb{R}^{n+2}$ respectively by:

$$\begin{array}{lll} M^n \subset & S^{n+1}(1) \subset & \mathbb{R}^{n+2} \\ \nabla \, \triangle, & \overline{\nabla} \, \overline{\triangle}, & \widetilde{\nabla} \, \widetilde{\triangle}. \end{array}$$

In order to facilitate the description, we state the following lemma in front of the proof of Theorem 1.1.

Lemma 2.1. *Let ξ be a unit vector field on $S^{n+1}(1)$ extended from a normal unit vector field of M^n, H be the mean curvature vector field of M^n. For functions $\mathcal{G}$ on $\mathbb{R}^{n+2}$, $G = \mathcal{G}|_{S^{n+1}}$ and $g = G|_{M^n}$, at any $x \in S^{n+1}(1)$ (as a position vector field) we have:*

$$\left\{ \begin{array}{ll} \widetilde{\triangle}\mathcal{G}|_{S^{n+1}} & = \overline{\triangle}G + nx(\mathcal{G}) + xx(\mathcal{G}) \\ \overline{\triangle}G|_{M^n} & = \triangle g - \xi(G)\langle H, \xi\rangle + \xi\xi(G) - \overline{\nabla}_{\xi}\xi(G) \end{array} \right. \tag{7}$$

Proof of Theorem 1.1. We take the first step by determining the eigenvalues corresponding to φ_i $(i = 1, 2, 3)$. Clearly, based on Lemma 2.1, a direct calculation depending on the minimality of M^n leads to

$$\triangle\varphi_1 = -n\varphi_1. \tag{8}$$

Besides, in conjunction with Codazzi equation, we get another straightforward result:

$$\triangle\varphi_3 = -|B|^2\varphi_3 = -(g-1)n\varphi_3 = -3n\varphi_3, \tag{9}$$

where B is the second fundamental form of M^n, and the second " $=$ " in (9) is an assertion of [PT]. According to Solomon [Sol], the eigenvalue corresponding to φ_2 is equal to $2n$. As a matter of fact, this conclusion can also be derived from a few basic facts and Lemma 2.1—some formulas in this process will be useful later:

It is well known that there exists a unique c_0 with $-1 < c_0 < 1$ such that the minimal isoparametric hypersurface M^n (of FKM-type) is given by $M^n = f^{-1}(c_0)$.

We can choose the unit normal vector field to be

$$\xi = \frac{\overline{\nabla} f}{|\overline{\nabla} f|}|_{M^n} = \frac{\widetilde{\nabla} F - 4Fx}{4\sqrt{1-F^2}}|_{M^n}.$$

Extending ξ along the normal geodesics such that $\overline{\nabla}_\xi \xi = 0$, it follows that

$$\xi(\varphi_2) = \langle \xi, \overline{\nabla}\varphi_2\rangle = \langle \frac{\widetilde{\nabla} F - 4Fx}{4\sqrt{1-F^2}}, 2Px - 2\varphi_2 x\rangle = -2\sqrt{\frac{1+f}{1-f}}\varphi_2, \tag{10}$$

and thus

$$\xi\xi(\varphi_2) = \langle \xi, \overline{\nabla}\xi(\varphi_2)\rangle = -4\varphi_2.$$

Here, we extended φ_2 to $S^{n+1}(1)$ and $\mathbb{R}^{n+2}$ in a natural way. Then combining with (7) and $H = 0$, we arrive at

$$\triangle\varphi_2 = -2n\varphi_2. \tag{11}$$

Next, we aim to investigate the critical sets of φ_i $(i = 1, 2, 3)$. Let $e_1, e_2, ..., e_n$ be an orthonormal tangent frame field on M^n with $A_\xi e_i = \mu_i e_i$ $(i = 1, 2, ..., n)$, where A_ξ is the shape operator. According to Münzner, the principal curvature $\mu_i \in \{\cot\theta_j = \cot(\theta_1 + \frac{j-1}{4}\pi) \mid 0 < \theta_1 < \frac{\pi}{4},\ j = 1, 2, 3, 4\}$.

(i) For each $e_i \in T_xM^n$, we have

$$\langle \nabla\varphi_1, e_i\rangle = e_i\langle x, q_1\rangle = \langle e_i, q_1\rangle. \tag{12}$$

It follows that x is a critical point of φ_1 if and only if $q_1 \in Span\{x, \xi(x)\}$. In other words, q_1 lies on some normal geodesic $v(t)$ $(-\pi \leq t \leq \pi)$ with $v(0) = x,\ v'(0) = \xi(x)$. Therefore the number of critical points of φ_1 is

$$\sharp C(\varphi_1) = \frac{2\pi}{\pi/g} = 2g = 8.$$

Here, we used the known fact that the distance between two focal submanifolds is equal to π/g (*cf.* [CR]). Furthermore, recall the formula of Hessian:

$$Hess(\varphi_1)_{ij} = \langle e_i, \nabla_{e_j}\nabla\varphi_1\rangle.$$

Restricted to a critical point x, using (12) we express it as

$$Hess(\varphi_1)|_x = -diag\{\ \langle \mu_1\xi - x, q_1\rangle,\ \langle \mu_2\xi - x, q_1\rangle, ...,\ \langle \mu_n\xi - x, q_1\rangle\ \}. \tag{13}$$

Writing $q_1 = \cos t\ x + \sin t\ \xi$ $(-\pi < t < \pi)$ for a fixed x, a direct calculation leads to

$$\begin{aligned}\langle \mu_i\xi - x, q_1\rangle = 0 \quad &\Leftrightarrow \quad \sin t(\cot\theta_i - \cot t) = 0\\ &\Leftrightarrow \quad q_1 \in M_+ \cup M_- \cup M^n.\end{aligned}$$

From the assumption $q_1 \in S^{n+1}(1)\backslash\{M_+, M_-, M^n\}$, we derive that φ_1 is a Morse function, as desired.

(ii) Similarly, for each $e_i \in T_xM^n$, we have

$$\langle \nabla\varphi_3, e_i\rangle = e_i\langle \xi, q_1\rangle = -\langle A_\xi e_i, q_1\rangle = -\langle \mu_i e_i, q_1\rangle.$$

Since $\mu_i \in \{\cot\theta_j = \cot(\theta_1 + \frac{j-1}{4}\pi) \mid 0 < \theta_1 < \frac{\pi}{4},\ j = 1,2,3,4\}$, it is easy to see that $\mu_i \neq 0\ \forall i$. Thus x is a critical point of φ_3 if and only if $q_1 \in Span\{x, \xi(x)\}$. Analogously,

$$\sharp C(\varphi_3) = \frac{2\pi}{\pi/g} = 2g = 8.$$

Furthermore, $Hess(\varphi_3)$ at a critical point x can be expressed as

$$Hess(\varphi_3)|_x = -diag\{\ \mu_1\langle \mu_1\xi - x, q_1\rangle,\ \mu_2\langle \mu_2\xi - x, q_1\rangle, ...,\ \mu_n\langle \mu_n\xi - x, q_1\rangle\ \}. \tag{14}$$

Again, our choice of q_1 guarantees that φ_3 is a Morse function.

(iii) From the formula (10), we derive that

$$\begin{aligned} \nabla\varphi_2 &= \widetilde{\nabla}\varphi_2 - x(\varphi_2)x - \xi(\varphi_2)\xi \\ &= 2(Px - \varphi_2 x + \varphi_2\sqrt{\frac{1+c_0}{1-c_0}}\xi). \end{aligned} \tag{15}$$

Immediately, a simple calculation shows that φ_2 satisfies

$$\begin{cases} |\nabla\varphi_2|^2 = 4(1 - \frac{2}{1-c_0}\varphi_2^2) \\ \triangle\varphi_2 = -2n\varphi_2. \end{cases} \tag{16}$$

By definition, φ_2 is an isoparametric function on M^n. Define the focal submanifolds by $N_\pm := \{x \in M^n \mid \varphi_2 = \pm\sqrt{\frac{1-c_0}{2}}\}$. Therefore the critical set of φ_2 is the union of its focal submanifolds:

$$C(\varphi_2) = N_+ \cup N_-.$$

We are now in a position to complete the proof of Theorem 1.1 by verifying Lemma 1.1.

Proof of Lemma 1.1. As indicated before, the focal submanifold M_+ of FKM-type is

$$M_+ := f^{-1}(+1) = \{x \in S^{n+1}(1) \mid \langle P_0x, x\rangle = \langle P_1x, x\rangle = \cdots = \langle P_mx, x\rangle = 0\}.$$

Define a map:

$$\begin{aligned} h_+ : M_+ &\to S^{n+1}(1) \\ x &\mapsto \cos t\ x + \sin t\ Px \end{aligned}$$

where $\cos t = \sqrt{\frac{1}{2}(1 + \sqrt{\frac{1+c_0}{2}})}$, $\sin t = \sqrt{\frac{1}{2}(1 - \sqrt{\frac{1+c_0}{2}})}$. It is easy to show that

$$\langle Ph_+(x), h_+(x)\rangle = \sqrt{\frac{1-c_0}{2}},\ \ i.e.\ h_+(x) \in N_+.$$

Thus the image of h_+ is contained in N_+. On the other hand, define another map:

$$\begin{aligned} j_+ : N_+ &\to M_+ \\ x &\mapsto \cos t\ x + \sin t\ \xi(x) \end{aligned}$$

with the same values of $\cos t$ and $\sin t$, and $\xi = \frac{\overline{\nabla} f}{|\overline{\nabla} f|}$. Evidently, j_+ is well defined and is just the inverse function of h_+. This means that the focal submanifold N_+ of φ_2 on M^n is diffeomorphic to the focal submanifold M_+ of f on $S^{n+1}(1)$.

We conclude the proof by investigating the mean curvatures of the level hypersurfaces $N_t := \varphi_2^{-1}(t)$, $t \in (-\sqrt{\frac{1-c_0}{2}}, \sqrt{\frac{1-c_0}{2}})$. Following the formula of the mean curvature $h(t)$ (*cf.* [GT2]), we have:

$$h(t) = \frac{b'(t) - 2a(t)}{2\sqrt{b(t)}} = \frac{n - \frac{4}{1-c_0}}{\sqrt{1 - \frac{2t^2}{1-c_0}}}\ t \tag{17}$$

Obviously, the isoparametric hypersurface $N_0 = \varphi_2^{-1}(0)$ is minimal in M^n. In addition, the minimality of M^n implies:

$$c_0 = \frac{m_- - m_+}{m_- + m_+} = \frac{l - 2m - 1}{l - 1}, \quad n = 2l - 2,$$

then we obtain that

$$n - \frac{4}{1 - c_0} = 0 \Leftrightarrow m = 1 \ (\ \textit{the improper case}\ (\textit{cf.}\ [GT2])).$$

In conclusion, in the improper case, all the level hypersurfaces of φ_2 are minimal!

The same argument applies to N_- with a little change of the values:

$$\cos t = \sqrt{\frac{1}{2}(1 + \sqrt{\frac{1 + c_0}{2}})}, \quad \sin t = -\sqrt{\frac{1}{2}(1 - \sqrt{\frac{1 + c_0}{2}})}.$$

□

The proof of Theorem 1.1 is now complete!

3. COUNTEREXAMPLES ON M_-

Proof of Theorem 1.2. Implementing the previous arguments in Section 2, it is not difficult to find that ω_2 on M_- is an eigenfunction corresponding to the eigenvalue $\dim M_- = l + m - 1$, and the number of its critical points is $\frac{2\pi}{2\pi/g} = g = 4$ (*cf.* [CR]). Therefore, in order to complete the proof of Theorem 1.2, we need only to confirm that ω_1 is an isoparametric function on M_- and prove Lemma 1.2.

Firstly, noticing the Euclidean gradient $\widetilde{\nabla}\omega_1$ can be expressed by

$$\widetilde{\nabla}\omega_1 = 2Px = 2\langle Px, x\rangle x + 2\Big(Px - \langle Px, x\rangle x\Big),$$

we claim that

Claim: $y := Px - \langle Px, x\rangle x \in T_xM_-$.

Holding this claim, it follows that $\nabla\omega_1 = 2y = 2(Px - \langle Px, x\rangle x)$. Then a simple calculation leads to

$$\begin{cases} |\nabla\omega_1|^2 = 4(1-\omega_1^2) \\ \triangle\omega_1 = -4m\omega_1, \end{cases} \tag{18}$$

where the second equality is due to Solomon [Sol]. Namely, ω_1 is an isoparametric function on M_-. Define the focal submanifolds of ω_1 by $V_\pm := \{x \in M_- \mid \omega_1 = \pm 1\}$. Then the critical set of ω_1 is

$$C(\omega_1) = V_+ \cup V_-.$$

Now we are left to prove the previous Claim and Lemma 1.2.

Proof of Claim. Firstly, we rewrite the focal submanifold

$$M_- := \{x \in S^{n+1}(1) \mid \sum_{\alpha=0}^{m} \langle P_\alpha x, x\rangle^2 = 1\}$$

as

$$M_- \quad = \quad \{x \in S^{n+1}(1) \mid x = \sum_{\alpha=0}^{m} \langle P_\alpha x, x\rangle P_\alpha x\}$$

Define $\mathcal{P} := \sum_{\alpha=0}^{m} \langle P_\alpha x, x\rangle P_\alpha$, then for each $x \in M_-$ we have

$$\mathcal{P} \in \Sigma \quad and \quad \mathcal{P}x = x. \tag{19}$$

Since $\mathcal{P}$ is an orthogonal symmetric matrix with vanishing trace, we can decompose $\mathbb{R}^{2l}$ as

$$\mathbb{R}^{2l} = E_+(\mathcal{P}) \oplus E_-(\mathcal{P}).$$

With respect to this decomposition, $2y \in \mathbb{R}^{2l}$ can be written as

$$2y = (y + \mathcal{P}y) + (y - \mathcal{P}y).$$

Denoting $P = \sum_{\beta=0}^{m} a_\beta P_\beta$ with $\sum_{\beta=0}^{m} a_\beta^2 = 1$, we have

$$\begin{aligned} y + \mathcal{P}y &= Px - \langle Px, x\rangle x + \mathcal{P}Px - \langle Px, x\rangle \mathcal{P}x \\ &= P\mathcal{P}x + \mathcal{P}Px - 2\langle Px, x\rangle x \\ &= \sum_{\beta=0}^{m} a_\beta P_\beta\Big(\sum_{\alpha=0}^{m} \langle P_\alpha x, x\rangle P_\alpha x\Big) + \sum_{\alpha=0}^{m} \langle P_\alpha x, x\rangle P_\alpha\Big(\sum_{\beta=0}^{m} a_\beta P_\beta x\Big) - 2\langle Px, x\rangle x \\ &= 2\sum_{\alpha=0}^{m} a_\alpha \langle P_\alpha x, x\rangle x - 2\sum_{\beta=0}^{m} a_\beta \langle P_\beta x, x\rangle x \\ &= 0, \end{aligned}$$

which leaves $2y = y - \mathcal{P}y$, *i.e.* $y \in E_-(\mathcal{P})$.

On the other hand, setting $y = Px - \langle Px, x\rangle x = Qx$, where

$$Q := P - \langle Px, x\rangle \mathcal{P} \in Span\{P_0, P_1, ..., P_m\},$$

it is easy to find that

$$\langle Q, \mathcal{P}\rangle = 0.$$

Comparing with (*cf.* [FKM])

$$T_x^{\perp} M_- = \{\nu \in E_-(\mathcal{P}) \mid \langle \nu, Qx\rangle = 0,\ \forall \langle Q, \mathcal{P}\rangle = 0\},$$

we get immediately the Claim. □

Now we are in a position to prove Lemma 1.2.

Proof of Lemma 1.2. Under an orthogonal transformation, we can express P as

$$P = T^t \begin{pmatrix} I_l & 0 \\ 0 & -I_l \end{pmatrix} T, \quad with\ T^tT = I_{2l}.$$

Write $Tx = (z, w) \in \mathbb{R}^l \times \mathbb{R}^l$ for $x \in S^{n+1}(1)$. The condition $\langle Px, x\rangle = 1$ is equivalent to

$$|z|^2 - |w|^2 = 1,$$

which implies $|z|^2 = 1,\ |w|^2 = 0$. On the other hand, we observe that

$$\begin{aligned} V_+ &:= \{x \in M_- \mid \langle Px, x\rangle = 1\} \\ &= \{x \in S^{2l-1} \mid \langle Px, x\rangle = 1\}. \end{aligned}$$

Thus we get an isometry

$$V_+ \underset{isom.}{\cong} S^{l-1}(1).$$

Similarly,

$$V_- \underset{isom.}{\cong} S^{l-1}(1).$$

Therefore,

$$\dim V_+ = \dim V_- = l - 1.$$

Now the proof of Theorem 1.2 is complete!

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